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CONJUGACY AND PRINCIPAL SOLUTION OF GENERALIZED HALF-LINEAR SECOND ORDER DIFFERENTIAL EQUATIONS

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Abstract. We study the generalized half-linear second order differential equation and the associated Riccati type differential equation. We introduce the concepts of minimal and principal solutions of these equations and using these concepts we prove a new conjugacy criterion for the generalized half-linear equation.

1. Introduction

We consider the differential equation of the form

$$\big(r(t)x'\big)' + c(t)f(x, r(t)x') = 0 \tag{1}$$

with continuous functions c, r and $r(t) > 0$, under assumptions on the function f which guarantee that the solution space of this equation is homogeneous, i.e., if x is a solution of (1) then λx, $\lambda \in \mathbb{R}$, is a solution as well. Particular assumptions on the function f will be listed later. Equation (1), under these assumptions, was investigated by Hungarian mathematicians I. Bihari [2, 3] and Á. Elbert [12, 13]. It was shown that many statements of the classical oscillation theory for the Sturm-Liouville linear differential equation

$$\big(r(t)x'\big)' + c(t)x = 0 \tag{2}$$

can be extended in a natural way to (1).

A typical model of (1) is the "classical" half-linear differential equation

$$\big(r(t)\Phi(x')\big)' + c(t)\Phi(x) = 0, \quad \Phi(x) := |x|^{p-2}x,\ p > 1, \tag{3}$$

which attracted considerable attention in recent years, see, e.g., [1, 10]. If q denotes the conjugate exponent of p, i.e., $\frac{1}{p} + \frac{1}{q} = 1$, the first term in (3) can be written as

$$\big(\Phi(r^{q-1}x')\big)' = (p-1)|r^{q-1}x'|^{p-2}(r^{q-1}x')'$$

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and hence (3) can be written as

$$\left(r^{q-1}(t)x'\right)' + \frac{c(t)}{p-1}\Phi(x)|r^{q-1}(t)x'|^{2-p} = 0$$

which is equation of the form (1). Recall also that an important role in the investigation of the qualitative properties of solutions of (3) is played by the Riccati type differential equation

$$w' + c(t) + (p-1)r^{1-q}(t)|w|^q = 0 \tag{4}$$

related to (3) by the Riccati substitution $w = r\Phi(x'/x)$.

Since 1987, when the last of the series of papers [2, 3, 12, 13] were published, the qualitative theory of (3) made a big progress and it is a natural question which results of this theory can be extended to a more general equation (1). In our paper we follow this idea. First, we introduce the concept of the minimal (and maximal) solution of the Riccati type differential equation associated with (1). Next, we define the concept of the principal solution of (1) and we use the properties of this solution to establish a conjugacy criterion for this equation. In the last part of the paper we formulate open problems for the next investigation.

2. Generalized Riccati type equation

We start this section with the assumptions on the function f in (1) which are taken from the papers [12, 13]. We also refer to [13] for a discussion concerning these assumptions.

(i) The function f is continuous on $\Omega = \mathbb{R} \times \mathbb{R}_0$, where $\mathbb{R}_0 = \mathbb{R} \setminus \{0\}$;
(ii) It holds $xf(x, y) > 0$ if $xy \neq 0$;
(iii) The function f is homogeneous, i.e., $f(\lambda x, \lambda y) = \lambda f(x, y)$ for $\lambda \in \mathbb{R}$ and $(x, y) \in \Omega$;
(iv) The function f is sufficiently smooth in order to ensure the continuous dependence and the uniqueness of solutions of the initial value problem $x(t_1) = x_0$, $x'(t_1) = x_1$ at some $(x_0, x_1) \in \Omega$;
(v) Let $F(t) := tf(t, 1)$, then

$$\int_{-\infty}^{\infty} \frac{dt}{1+F(t)} < \infty \quad \text{and} \quad \lim_{|t|\to\infty} F(t) = \infty. \tag{5}$$

Let g be the differentiable function given by the formula

$$g(u) = \begin{cases} \int_{1/u}^{\infty} \frac{ds}{F(s)} & \text{if } u > 0, \\ -\int_{-\infty}^{1/u} \frac{ds}{F(s)} & \text{if } u < 0, \end{cases} \tag{6}$$

and $g(0)=0$. Then g is strictly increasing and $\lim_{u\to\pm\infty} g(u)=\pm\infty$. If x is a solution of (1) such that $x(t)\neq 0$, then the function $v=g(rx'/x)$ solves the Riccati type differential equation

$$v'+c(t)+r^{-1}(t)H(v)=0, \tag{7}$$

where the function H is given by the formula

$$H(v)=[g^{-1}(v)]^2 g'(g^{-1}(v)) \tag{8}$$

with $H(0)=0$ (g^{-1} being the inverse function of g). Conversely, having a function $H(v)>0$ for $v\neq 0$, with $H(0)=0$, such that

$$\int_{-\infty}^{-1}\frac{ds}{H(s)}<\infty, \quad \int_{1}^{\infty}\frac{ds}{H(s)}<\infty, \tag{9}$$

one can associate with (7) an equation (1) with f satisfying (i) – (v) as follows. The function g is given as the solution of the differential equation

$$g'(u)=\frac{1}{u^2}H(g(u)), \quad g(0)=0,$$

and the function $f:\mathbb{R}\times\mathbb{R}_0\to\mathbb{R}$ is given by the formula

$$f(1,u):=\frac{1}{g'(u)}, \quad f(t,s):=\begin{cases} tf(1,t/s), & t\neq 0,\\ 0 & t=0.\end{cases} \tag{10}$$

Let us stop for a moment by the assumption (iii). This assumption was introduced by Bihari [2, 3] and modified later by Elbert [13] in such a way that the equality $f(\lambda x,\lambda y)=\lambda f(x,y)$ is supposed only for $\lambda>0$. Under this weaker version of homogeneity assumption, one obtains *two* Riccati type equations of the form (7), one for the ratio rx'/x with $x(t)>0$ and the other one for $x(t)<0$. However, as noted in [13], both Riccati equations can be treated in the same manner, so we adopt here the original Bihari's assumption (iii).

Following [13], to study oscillatory properties of (1) in more details, we also need the following assumption:

(vi) The function H given by (8) is strictly *convex*.

This assumption is satisfied e.g. when the function $\log F(t)$ is strictly concave, see [13]. Under this assumption, the function H is decreasing for $u\leq 0$ and increasing for $u\geq 0$. We denote by H_-^{-1}, H_+^{-1} the inverse functions of H restricted to nonpositive and nonnegative half-line, respectively. Also, H is a locally Lipschitz function under assumption (vi), which means the the initial value problem associated with (7) is uniquely solvable and hence graphs of solutions of (7) cannot intersect.

Another motivation for the investigation of the generalized Riccati equation is the so-called "perturbation method" which in half-linear oscillation theory has been introduced in [15] and further developed in [10, Section 5.2]. Consider equation (3) with $r(t) \equiv 1$ as a perturbation of the Euler half-linear differential equation with the so-called critical constant

$$(\Phi(x'))' + \frac{\gamma_p}{t^p}\Phi(x) = 0, \quad \gamma_p := \left(\frac{p-1}{p}\right)^p. \tag{11}$$

This equation is known to be nonoscillatory and possesses a solution $h(t) = t^{\frac{p-1}{p}}$. The Riccati equation associated with (11) is

$$w' + \frac{\gamma_p}{t^p} + (p-1)|w|^q = 0. \tag{12}$$

The function $v := h^p(w - w_h)$, where w is a solution of (4) with $r = 1$ and $w_h = \Phi(h'/h) = \left(\frac{p-1}{p}\right)^{p-1} t^{1-p}$, is a solution of the equation

$$v' + \left(c(t) - \frac{\gamma_p}{t^p}\right)t^{p-1} + (p-1)t^{-1}\left[|v+\Gamma|^q - v - \Gamma^q\right] = 0, \quad \Gamma := \left(\frac{p-1}{p}\right)^{p-1}. \tag{13}$$

The last equation is just the Riccati type equation of the form (7) since the function

$$H(v) = |v+\Gamma|^q - v - \Gamma^q \tag{14}$$

satisfies all assumptions on the function H given above.

Finally, at the end of this section, note that a closer examination of our treatment reveals that we can more or less forget about the original second order differential equation (1) and consider directly the equation (7) with a strictly convex function H satisfying $H(0) = \min_{u\in\mathbb{R}} H(u) = 0$ such that (9) holds.

3. Minimal, maximal, and principal solutions

First we introduce the concept of the minimal (and maximal) solution of (7). We modify the construction given in [21, Sec. 15], see also [4].

Suppose that (7) possesses a solution which is defined on some interval $[T, \infty)$ (such a solution we call *proper*). Define

$$\mathcal{Y} = \{y \in \mathbb{R}, \text{ the solution } v \text{ of (7) given by } v(T) = y \text{ is proper}\}$$

and let

$$\tilde{y} = \inf \mathcal{Y}. \tag{15}$$

Denote by $v_{\min}$ the solution of (7) given by the initial condition $v_{\min}(T) = \tilde{y}$. This solution we call the *minimal solution* (at ∞).

Next we show that the minimal solution is well defined.

Lemma 1. *The set $\mathcal{Y}$ is bounded from below. In particular, consider an interval $[T, T+\tau]$, where $\tau > 0$ is arbitrary. There exists $y_0 \in \mathbb{R}$ such that any solution of* (7) *with $v(T) < y_0$ satisfies*

$$\lim_{t \to t_1-} v(t) = -\infty$$

for some $t_1 \in [T, T+\tau]$.

Proof. Denote

$$C = \min_{[T,T+\tau]} c(t), \quad R = \max_{[T,T+T]} r(t),$$

and together with (7), consider the equation with constant coefficients

$$u' + C + \frac{H(u)}{R} = 0. \tag{16}$$

Then by the standard theorem for differential inequalities (see, e.g., [18]), if $v(T) < u(T)$, then $v(t) < u(t)$ for $t > T$ for which $v(t)$ exists.

Now consider equation (16). We have

$$\int_{u(T)}^{u(t)} \frac{ds}{-C - R^{-1}H(s)} = t - T.$$

For $u \to -\infty$ we have $H(u) \to \infty$ and hence there exists $\tilde{u}$ such that $-C - R^{-1}H(u) < 0$ for $u < \tilde{u}$, i.e., $u(t)$ is decreasing and

$$\int_{u(t)}^{u(T)} \frac{ds}{C + R^{-1}H(s)} = t - T$$

if $u(T) < \tilde{u}$. Hence, by (9),

$$\infty > \int_{-\infty}^{u(T)} \frac{ds}{C + R^{-1}H(s)} > \int_{u(t)}^{u(T)} \frac{ds}{C + R^{-1}H(s)} = t - T.$$

Now, if $u(T) \to -\infty$, the first integral in the previous formula tends to 0, which means that $t \to T$, i.e., $t - T < \tau$ for $u(T)$ sufficiently negative. Hence $u(t)$ has to blow down to $-\infty$ inside of the interval $[T, T+\tau]$ and inequality for solutions of (7) and (16) implies that a solution v of (7) starting with sufficiently negative initial value $v(T)$ has the same property. $\square$

Next we show that the minimal solution $v_{\min}$ is really proper.

Lemma 2. *The minimal solution $v_{\min}$ of* (7) *is a proper solution.*

Proof. By contradiction, suppose that $v_{\min}$ is not proper, i.e., $v_{\min}(t_1-) = -\infty$ for some $t_1 > T$. Let $t_2 > t_1$ be arbitrary. Recall that we suppose that (7) possesses a solution v defined in the whole interval $[T, \infty)$ and that by definition of $v_{\min}$ it holds $v(T) > \tilde{y}$, where $\tilde{y}$ is given by (15). Let $\hat{x}$ be the solution of (1) given by the initial condition $\hat{x}(t_2) = 0$, $r(t_2)\hat{x}'(t_2) = -1$ and let $\hat{v} = g(r\hat{x}'/\hat{x})$ be the associated solution of (7). Then $\hat{v}(t_2-) = -\infty$ and by the unique solvability of (7)

$$v(t) > \hat{v}(t) > v_{\min}(t), \quad \text{for } t \in [T, t_1).$$

But this inequality contradicts the definition of $\tilde{y}$, so the solution $v_{\min}$ is proper. □

Similarly we define the *maximal solution* (in the neighbourhood of $-\infty$) of (7). We suppose that (7) possesses a solution defined in an interval $(-\infty, A]$ (such a solution we call again *proper* at ∞) and we denote

$$\mathcal{Z} = \{z \in \mathbb{R}, \text{ the solution } v \text{ of (7) given by } v(A) = z \text{ is proper at } \infty\},$$

and $\tilde{z} = \sup \mathcal{Z}$. The maximal solution in the neighbourhood of $-\infty$ is the solution given by the initial condition $v_{\max}(A) = \tilde{z}$. The proofs that the set $\mathcal{Z}$ is nonempty and bounded above and that the solution $v_{\max}$ is proper are the same as in case of the minimal solution.

Definition 1. Suppose that (1) is nonoscillatory, i.e., there exists a solution of this equation which is nonzero in an interval $[T, \infty)$, and let $v_{\min}$ be the minimal solution of the associated Riccati equation (7). The *principal solution* of (1) at ∞ is the solution given by the formula

$$x(t) = C \exp\left\{ \int_T^t \frac{g^{-1}(v_{\min}(s))}{r(s)}\, ds \right\}, \tag{17}$$

where $C \neq 0$ is a real constant. The principal solution is determined uniquely up to a nonzero multiplicative factor.

The principal solution at $-\infty$ is defined via the maximal solution of (7) in the neighbourhood of $-\infty$ analogously.

Next we present a Sturmian type comparison theorem for minimal solution of (7). This statement can be regarded as a complement of [12, Theorem 4.10]. A similar statement holds for maximal solutions.

Theorem 1. *Together with* (1) *consider the equation of the same form*

$$\big(\hat{r}(t)x\big)' + \hat{c}(t) f(x, \hat{r}(t)x') = 0 \tag{18}$$

with the continuous functions $\hat{c}$, $\hat{r}$ satisfying

$$\hat{c}(t) \geq c(t), \quad 0 < \hat{r}(t) \leq r(t) \tag{19}$$

for large t. Suppose that (18) *is nonoscillatory and denote by $v_{\min}$, $\hat{v}_{\min}$ minimal solutions of* (7) *and of the Riccati equation associated with* (18)*, respectively. Then $\hat{v}_{\min}(t) \geq v_{\min}(t)$ in the common interval of their existence.*

Proof. First of all, note that nonoscillation of (18) implies nonoscillation of (1) by the Sturmian theorem for (1), see [13]. Our proof follows the idea of [4, Theorem 2]. Let u be a proper solution of the equation

$$u' + \hat{c}(t) + \hat{r}^{-1}(t)H(u) = 0 \tag{20}$$

(which is the Riccati equation associated with (18)), i.e., a solution which is defined on some interval $[T, \infty)$. Consider the solution v of (7) given by the initial condition $v(T) = u(T)$. Then inequalities between c, $\hat{c}$, r, and $\hat{r}$ imply that $v(t) \geq u(t)$ for $t \geq T$. Now, by contradiction, suppose that the minimal solutions $v_{\min}$, $\hat{v}_{\min}$ satisfy $v_{\min}(t_1) > \hat{v}_{\min}(t_1)$ at some $t_1 > T$. Consider the solution v of (7) given by $v(t_1) = \hat{v}_{\min}(t_1)$. Then by the same argument as in the previous part of the proof $v(t) \geq \hat{v}_{\min}(t)$ for $t \geq t_1$. At the same time, since $v(T) < v_{\min}(T)$ we have $v(t) < v_{\min}(t)$ for $t \geq T$. This means that we have found a proper solution v of (7) which is less then the minimal solution of this equation, a contradiction. □

The next theorem shows that the principal solution has the property which is called *zero maximal property* in the linear case, see [19]. In the "linear terminology", it states that the largest zero point of the principal solution at ∞ is something like the left conjugate point of ∞.

Theorem 2. *Suppose that* (1) *is nonoscillatory, $\tilde{x}$ is its principal solution at ∞, and let T be its largest zero. Then any other solution of* (1) *has a zero in $[T, \infty)$.*

Proof. Suppose, by contradiction, that there is a solution x of (1) having no zero in $[T, \infty)$ and $v = g(rx'/x)$ is the associated solution of (7). Further, let $v_{\min}$ be the minimal solution of (7), i.e., $v_{\min} = g(r\tilde{x}'/\tilde{x})$. Then by the definition of the minimal solution we have $v(t) \geq v_{\min}(t)$ for large t. But this contradicts the fact that $v_{\min}(T+) = \infty$ while $v(T)$ is a real number, so the graphs of v and $v_{\min}$ have to intersect somewhere in (T, ∞) and this is impossible due to the unique solvability of (7). □

4. A conjugacy criterion

To make the results of this section better understandable, consider equation (3) for $t \in \mathbb{R}$. Suppose that

$$\int^{\infty} r^{1-q}(t)\, dt = \infty = \int_{-\infty} r^{1-q}(t)\, dt. \tag{21}$$

Under this assumption, $x(t) = 1$ is the only solution (up to a nonzero multiplicative factor) of the one-term equation

$$\big(r(t)\Phi(x')\big)' = 0 \tag{22}$$

which has no zero in $\mathbb{R}$ and this also means that the principal solutions at $-\infty$ and ∞ coincide. Note that in the terminology of the linear Sturm-Liouville differential equations (2), an equation whose principal solutions at $-\infty$ and ∞ coincide is said to be *1-special* on $\mathbb{R}$ (see [5]), while in the terminology of [17] the operator on the left-hand side of (22) is called *critical* on $\mathbb{R}$ in this case.

Now, if

$$\int_{-\infty}^{\infty} c(t)\, dt > 0, \tag{23}$$

i.e., we perturb one-term equation (22) (with r satisfying (21)) by a potential c with a positive mean value over $\mathbb{R}$, then equation (3) is *conjugate* in $\mathbb{R}$, i.e., there is a solution of (3) with at least two zeros in $\mathbb{R}$, see [22].

As a main result of this section we show that a similar statement holds for perturbations of the equation $(r(t)x')' = 0$ by the term $c(t)f(x, rx')$ with the potential c having a positive mean value over $\mathbb{R}$. This statement can be regarded as a "conjugacy complement" of the oscillation criterion [13, p. 240].

Theorem 3. *Suppose that*

$$\int_{-\infty} r^{-1}(t)\, dt = \infty = \int^{\infty} r^{-1}(t)\, dt \tag{24}$$

and that (23) *holds. Then* (1) *is conjugate in* $\mathbb{R}$*, i.e., there exists a solution of this equation with at least two zeros in* $\mathbb{R}$*.*

Proof. Our proof is completely different from that of [22] (for (3)) which is based on the so-called variational principle. It is an open problem whether this method can be extended to (1), we discuss this topic in the last section of this paper.

Condition (23) implies that there exists $T \in \mathbb{R}$ such that

$$\int_{T}^{\infty} c(t)\, dt > 0 \quad \text{and} \quad \int_{-\infty}^{T} c(t)\, dt > 0. \tag{25}$$

Let x be the solution of (1) given by the initial condition $x(T) = 1$, $x'(T) = 0$ and $v = g(rx'/x)$ be the associated solution of (7), i.e., this solution satisfies the initial condition $v(T) = 0$. Integrating (7) from T to t we get

$$v(t) + \int_{T}^{t} c(s)\, ds + \int_{T}^{t} r^{-1}(s)H(v(s))\, ds = 0.$$

Conditions (25) imply that there exist $\varepsilon > 0$ and $T_1 > T$ such that

$$v(t) + \varepsilon + \int_T^t r^{-1}(s) H(v(s))\, ds \le 0 \quad \text{for } t \ge T_1. \tag{26}$$

Let

$$S(t) = -\varepsilon - \int_T^t r^{-1}(s) H(v(s))\, ds. \tag{27}$$

Then $S' = -r^{-1}H(v)$, i.e., $H_-^{-1}(-rS') = v$, where H_-^{-1} is the inverse function to H for $v \le 0$. Inequality (26) can be written as $v(t) \le S(t)$, i.e.,

$$H^{-1}(-S'(t)r(t)) \le S(t) < 0.$$

This implies, taking into account that H is decreasing for negative arguments,

$$-S'(t)r(t) \ge H(S(t)), \quad t \ge T_1,$$

and therefore

$$-\frac{S'(t)}{H(S(t)} \ge \frac{1}{r(t)}.$$

Integrating the last inequality from T_1 to t, $t > T_1$, we obtain

$$\int_{S(t)}^{S(T_1)} \frac{dv}{H(v)} \ge \int_{T_1}^t r^{-1}(s)\, ds.$$

However, this inequality shows that the solution v cannot be defined on the whole interval $[T, \infty)$ since the left-hand side is bounded in view the fact that the improper integral $\int_{-\infty} \frac{dv}{H(v)}$ is convergent, while the integral on the right-hand side tends to ∞ as $t \to \infty$ by (24). Hence the solution x has a zero point at some $t > T$. In the same way we prove that this solution must have a zero also for $t < T$, i.e., (1) possesses a solution with at least two zeros in $\mathbb{R}$. □

Remark 1. In the previous statement we have considered (1) as a perturbation of the *one-term* equation $(r(t)x')' = 0$. Motivated by the linear case (2) and the classical half-linear case (3), we conjecture that Theorem 3 can be formulated in the following more general setting.

Conjecture 1. *Suppose that the equation*

$$\big(r(t)x'\big)' + \tilde{c}(t) f(x, r(t)x') = 0, \tag{28}$$

with a continuous function $\tilde{c}$, is disconjugate in $\mathbb{R}$ and its principal solutions at $\pm\infty$ coincide (i.e., this equation is critical or 1-special in the above mentioned terminology). Denote by $\tilde{x}$ this simultaneous principal solution at $\pm\infty$. If

(29) $$\int_{-\infty}^{\infty} \big[c(t) - \tilde{c}(t)\big] h(\tilde{x}(t))\, dt > 0,$$

then (1) *is conjugate in* $\mathbb{R}$.

In (29), an unknown function h appears, which is $h(x) = |x|^p$ in the classical half-linear case (3), see [8]. It is not clear at this moment which function is its appropriate substitution in the general half-linear case (1). This problem is closely connected with the Picone type identity which we discuss in the next section.

5. Comments and open problems

In this section we discuss some open problems associated with our investigation.

(i) The first problem concerns the so-called conditional oscillation. To explain it, consider equation (1) with $r(t) = 1$, i.e., the equation

(30) $$x'' + c(t) f(x, x') = 0.$$

Following the linear and classical half-linear case, equation (30) is said to be *conditionally oscillatory* if there exists a constant $\lambda_0 > 0$ such that (30) with $\lambda c(t)$ instead of $c(t)$ is oscillatory for $\lambda > \lambda_0$ and nonoscillatory for $\lambda < \lambda_0$. The function c is called *conditionally oscillatory potential* in this cases. Conditionally oscillatory potentials play an important role in the oscillation theory since they represent, in a certain sense, a borderline between oscillation and nonoscillation. In the linear case $f(x, x') = x$, and hence $H(v) = v^2$ in the associated Riccati equation, it is known that the conditionally oscillatory potential is $c(t) = t^{-2}$ and the oscillation constant is $\lambda_0 = \frac{1}{4}$. In the classical half-linear case (3) we have $H(v) = (p-1)|v|^q$ in the associated Riccati equation, q being the conjugate number to p. In this case $c(t) = t^{-p}$ is the conditionally oscillatory potential with the oscillation constant $\lambda_0 = \left(\frac{p-1}{p}\right)^p$. Finally, consider the equation

(31) $$v' + d(t) + (p-1)H(v) = 0, \quad H(v) = |v + \Gamma|^q - v - \Gamma^q$$

with a continuous function d. An equation of this form we get from (13) via the transformation of independent variable $t \to \log t$. The conditionally oscillatory potential in (31) is again t^{-2}. Observe that in case of linear equation and equation (31) we have $H(v) \sim Kv^2$ for $v \to 0$ (for some $K \in \mathbb{R}$), while in case of the Riccati equation (4) associated with (3) we have $H(v) = (p-1)|v|^q$. This suggests the following conjecture.

Conjecture 2. *Suppose that*

(32) $$\lim_{v \to 0} \frac{H(v)}{|v|^\alpha} = L \in (0, \infty)$$

for some $\alpha > 1$. Then the conditionally oscillatory potential of (30) *is* $c(t) = t^{-\beta}$, β *being the conjugate number of* α, *i.e.* $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, *and the oscillation constant can be reconstructed from the value of the limit* L *in* (32).

(ii) The second problem is associated with the integral characterization of the principal solution of (1). In the linear case, it is known that a nonoscillatory solution x of (2) is principal at ∞ if and only if

$$\int^{\infty} \frac{dt}{r(t)x^2(t)} = \infty, \tag{33}$$

see [18, p. 355]. In the classical half-linear case (3), a similar equivalent characterization is missing, but several "candidates" for such a characterization have been suggested, see [6, 7, 9, 16] and the references given therein. A typical result along this line is the so-called *Mirzov integral condition*, see [20, 21]. It states that there exist positive real numbers $m_* \le m^*$ (defined as global extrema of a certain nonlinear function associated with the function $H(v) = |v|^q$), such that the following implications hold. If $\int^{\infty} r^{1-q}(t)|x(t)|^{-m_*}dt = \infty$ for a nonoscillatory solution of (3), then this solution is principal. Conversely, if x is the principal solution of (3) then $\int^{\infty} r^{1-q}(t)|x(t)|^{-m^*}dt = \infty$. We conjecture that a statement of this kind can be extended to (1) with the numbers m_*, m^* defined via the function H from (7) similarly as in the classical half-linear case. Note that in the linear case (2) we have $m_* = m^* = 2$, so Mirzov's integrals reduce to (33).

(iii) Another problem associated with the principal solution of (1) is connected with its definition. In [14], the principal solution of (3) is defined directly for this equation and not indirectly, via the minimal solution of the associated Riccati equation, as presented in Section 3. This construction is based on the half-linear Prüfer transformation where the so-called half-linear trigonometric functions appear. The generalized Prüfer transformation for (1) is established in [12] and we hope to use it to an alternative construction of the principal solution of (1) in a subsequent paper.

(iv) We finish this section with a problem which is connected with Picone's identity for linear and half-linear equations. If y is any differentiable function and w is a solution of the Riccati equation (4) associated with (3), then by a direct computation we have the identity

$$\left[|y|^p w\right]' = r|y'|^p - c|y|^p - (p-1)r^{1-q}P(r^{1/p}y', w\Phi(y)), \tag{34}$$

where

$$P(u,v) = \frac{|u|^p}{p} - uv + \frac{|v|^q}{q} \ge 0 \tag{35}$$

with equality in (35) if and only if $v = \Phi(u)$. The fact that the function P is nonnegative follows from the general Fenchel inequality

$$f(u) - uv + f^*(v) \geq 0, \tag{36}$$

$f^*(v) = \sup_u [uv - f(u)]$ being the Fenchel conjugate function to f, with the pair of mutually conjugate functions $f(u) = |u|^p/p$, $f^*(v) = |v|^q/q$. A natural question is whether some analogical identity to (34) can be established with a solution of the Riccati equation (7), where the conjugate function H^* to H appears. In particular, it is not clear at this moment which function in general case plays the role of the function $|y|^p$ in (34), see also the remark below Theorem 3. Picone's identity also opens the problem under which additional assumptions on the function f in (1) this equation has a variational structure, i.e., one can associate with this equation an energy functional similarly as the functional involving the term $r|y'|^p - c|y|^p$ which appears in (34). A solution of this problem could show for which equations of the form (1) one can use the so-called variational principle in their oscillation theory.

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